Pre-Service Teachers' Responses for Ratio and Proportion Items

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Proportional reasoning is important for informed decisions in proportional problem situations. This paper reports on mathematical content knowledge related to proportional reasoning of second-year, pre-service teachers. Responses to two ratio items provide insights into their correct method of solutions and common misconceptions. Anchor Points (Tatto, Peck, Schwille, Bankov, et al., 2012) were used to score the items. Some pre-service teachers had difficult with a scale ratio item but many more were unable to correctly respond to a proportional item.

Competency in proportional reasoning is becoming increasingly important for functioning in today's world, such as calculating best buys at the supermarket or choosing a mobile phone plan. It is the reasoning related to solving problems involving ratios and proportions. "A single proportion is a relationship between two quantities such that if you increase the size of one by a factor of *a*, then the other's measure must increase by the same factor to maintain the relationship" (Thompson & Saldanha, 2003, p.114).

This paper reports on second-year, pre-service teachers' MCK related to proportional reasoning as seen in their responses to two ratio items on a test of mathematical competency during their primary education program. Careful consideration of the pre-service teachers' responses to these ratio items provides insights into their correct methods of solution and common misconceptions that inform subsequent instruction.

This paper begins with a discussion of previous research in this area. This is followed by the methodology, then the pre-service teachers' responses are presented. Next, the discussion considers the common issues and proportional reasoning strategies revealed in their responses, including the degree of difficulty of the items as identified by Anchor Points (Tatto, et al., 2012). The analysis of correct methods and common incorrect responses for both of the ratio and proportion items identifies concerns of understanding of proportional reasoning strategies demonstrated by some second-year, pre-service teachers.

Background

Expertise in analysing the mathematics of change, such as "the relations between varying quantities [proportion] and the accumulation of those quantities" (Kaput, 1995, p. 49) has become important for all. Ben-Chaim, Fey, Fitzgerald, Benedetto and Miller (1998) express proportion as "a statement of the equality of two ratios" (p. 249). Proportional reasoning is the thinking involved when working with ratios (Reys, Lindquist, Lambdin, Smith, et al., 2012). Suggate, Davis, and Goulding (2006) claimed common mistakes in ratio and proportion are often the result of a focus on constant difference when multiplicative comparisons are more appropriate. The recent Teacher Education and Development Study in Mathematics (TEDS-M) international study of pre-service primary and secondary teachers from seventeen countries identified proportional reasoning as problematic for pre-service teachers when responding to test items (Tatto, et al., 2012). However, studies of pre-service teachers do not often include ratio and have focused on other areas of number knowledge (Southwell & Penglase, 2005).

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Since proportional reasoning has been noted to be difficult for students, researchers have trialled various studies. For example, Singh (2000) suggested posing problems, which encourage the construction of unit co-ordination schemes, such as supermarket best-buys. He advised that a student's experience with ratio and proportion should not be ignored by a focus on teaching algorithms and techniques disconnected from student's everyday experiences. Within the scope and sequence of the Australian Mathematics Curriculum, students from Year 1 to Year 10 also learn financial mathematics as an important context for the application of number and algebra (Australian Curriculum Assessment and Reporting Authority [ACARA], 2012).

A robust conception of proportion should be transferable to a variety of contexts, for example, density, speed and slope. Fassoulopoulos, Kariotoglou and Koumaras (2003) reported that students found it difficult to think about both of the variables, weight and volume, and the ratio, which forms density. Walter and Gerson (2007) described the emergence of connections in teachers' thinking between the notions of "additive structure, recursive linear equations, proportional relationships in discrete measurements, graphing, rise-over-run, data tables, and an embodied sense of slope as steepness of a mountainside" (p. 227). They claimed that an understanding of slope based on the calculation of the ratio 'rise over run' limits the development of connections between slope and ratio. This is consistent with Thompson's (1994) claim that calculation of speed from the formula limits the development of speed as a ratio.

Shulman (1987) wrote about the significant difference between content knowledge, that is, knowledge of the content of instruction and pedagogical content knowledge, that is, knowledge on how the content may be taught. So it is important that teachers' content knowledge is correct and complete. Thompson and Thompson (1994) claimed that a teacher's difficulty in helping his student thinking about speed was due to the teacher's difficulty in explaining his own understanding, possibly due to his conceptions having a calculation orientation and being thought about in terms of numbers, operations and procedures. They asserted that a calculation orientation held by teachers inhibits a conceptual orientation to the teaching of mathematics. Tall (1997) suggested that students' introduction to a mathematical concept should be qualitative and global, followed by more formal description of concepts. The successful application of the rules does not indicate a conceptual understanding.

Methodology

This study employed a mixed methods approach. The empirical component comprised pre-service teachers' test scores while their written responses provided a rich data source that was interpreted for their understandings and methods of solution.

The pre-service teachers in this study were a convenience sample and completing their second-year of a Bachelor of Education program at an Australian university. At the time of this study the pre-service teachers had completed half their program, including three units of study related to primary mathematics education and 47 school experience days participating in a primary classroom.

The test instrument was designed by the primary mathematics education unit co-ordinator and used as part of the program to assess pre-service teachers' mathematical skills and knowledge. This pencil and paper test was given under exam conditions over two hours and no calculators were permitted. The content ranged in difficulty, and up to a Year 8 level of mathematics (ACARA, 2012). There were 49 items including the following topics: number, fractions, decimals, percentage, ratio, geometry, area, volume, measurement, statistics and probability. Most items required short answers using words or symbols (numbers) and working out was encouraged.

The second-year pre-service teachers' written responses to the completed test papers (N=47) were used as data for this paper. Two ratio items were selected in order to examine their correct method of solutions and misconceptions of proportional reasoning (Table 1). These items were selected because of the detail recorded in their responses, which could be used for describing the different methods of solutions.

Table 1
Rate and ratio items and descriptions of sub-strands of Number and Algebra (ACARA, 2012)

Item No.	Year Two Item	Australian Curriculum, sub-strand:	Type of situation
	Comparison of liquid		
12	A cordial drink needs to be made up of syrup and water in the ratio 1:3 if you make enough cordial for 4 glasses, each containing 200 ml, how much syrup would you need for this?	Recognise and solve problems involving simple ratios (Year 7)	Proportion (part-whole)
13	**Comparison of length**		
	A toy car is made to scale: 1:40 has a length of 8 centimetres. What is the length of the full-sized car?	Recognise and solve problems involving simple ratios (Year 7)	Scaling (whole-whole)

For each of the problems shown in Table 1, there could be a number of strategies used to calculate the correct responses (and examples of these are discussed later). The comparison of liquid (Table 1, Item 12) was a proportion problem involving part-whole and multiplicative thinking. This would be expected knowledge at Year 7 as students recognise and solve problems involving simple ratios (ACARA, 2012). For Item 12, the pre-service teachers had to interpret the relationship of cordial in a drink, with one part syrup for three parts water when the whole drink was 200ml. They then had to calculate the proportion of cordial in ml required for four drinks. The correct solution was 200 ml.

The second problem in Table 1, the comparison of length problem, Item 13, was a simple scale problem and would involve whole-whole thinking. This would also be expected knowledge at Year 7 (ACARA, 2012). For this item, the pre-service teachers had to compare two separate wholes. The scale of the toy car is 1:40 and represents 1 cm on the toy car and 40 cm of the full sized car. The toy car was 8 cm and the scale increased by eight times and the correct response for the full sized car was 360 cm or 3 metres and 60 cm.

Scoring of items

Descriptive and quantitative methods were used to analyse the second-year, pre-service teachers' responses for both test items (Table 1). As part of the TEDS-M study of pre-service teachers' MCK Anchor Points were used as "descriptions of the performance of those future teachers who had scores at specific points on the scale" (Tatto et al., 2012, p. 136). The researchers chose to use these Anchor Points for scoring the level of difficulty for the different ratio items in this paper. Anchor Point 1 represented the lower level of MCK or a probability of 0.50 when correctly responding to an item and Anchor Point 2 represented a higher level of MCK or a probability of 0.30 when correctly responding to an item (Tatto, et al., 2012). For this paper the pre-service teachers' total percentage of correct responses for both items was scored at or below Anchor Point 1, at or below Anchor Point 2 or higher than Anchor Point 2.

All responses were recorded into an excel spread sheet for analysis and sorted according to correct responses, the same incorrect responses, various other responses, blank for no r esponse recorded. This also enabled the scoring of the Anchor Points and location of these two items with respect to them. In the next section, the results are grouped and include the discussion of responses and pre-service teachers' methods of solutions of both items.

Results and discussion

The number of correct responses received for the two ratio and proportion items are recorded in Table 2. It shows the percentage of correct responses as well as the level of difficulty as indicated by the Anchor Points (Tatto et al., 2012) for each item.

Table 2
Correct response rate for test items

Item No.	Item as appeared on test	Test 1 % correct 1st year (N=47)	Anchor Point (AP)
12	Comparison of liquid	24 (51%)	At AP 1
13	Comparison of length	41 (87%)	Below AP 1

In Table 2, Item 12, comparison of liquid was the more difficult item of the two items reported here since about half (51%) of the pre-service teachers responded correctly and so was scored at Anchor Point 1. Item 13, comparison of length was an easier item as more (81%) pre-service teachers correctly responded to this item. Item 13 was scored below Anchor Point 1 and it was likely more pre-service teachers would correctly respond to this item. Neither item was likely to be difficult for second-year pre-service teachers to answer correctly or scored at Anchor Point 2.

These results suggest that a second-year, pre-service teacher would be more likely to correctly answer proportion (Singh, 2000) and scale items, demonstrating understanding of proportion for part-whole and scale for whole-whole.

Table 3 shows common incorrect responses that were identified and appear in with the relative percentages for each incorrect answer ('n' refers to the number of incorrect responses for that particular item and cohort). These most common errors or misconceptions are discussed below.

Table 3

Common incorrect responses and relative percentages for Item 12 and Item 13

Incorrect responses for Item 12	% incorrect responses (n=23)	Incorrect responses for Item 13	% incorrect responses (n=6)
4 ml	2 (4%)		
50 ml	2 (4%)		
Various other	14 (30%)	Various other	4 (9%)
Blank	5 (11%)	Blank	2 (4%)

Item 12 comparison of liquid

In Table 3, the errors for Item 12 demonstrated by pre-service teachers were scored into common groupings of misconceptions. Two (4%) pre-service teachers possibly ignored the 200 ml and misinterpreted the worded problem. One pre-service teacher recorded the ratio of 1:3 as being 1 ml of syrup to 3 parts water and possibly was thinking 4 glasses required 4 ml of cordial drink.

For the response of 50 ml, another two (4%) pre-service teachers recorded their working out and incorrect thinking. For these incorrect responses, the first response showed 1:3 was the same ratio as 50:150 but then they recorded 50:150 = 200[ml] and chose 50ml for their response, and likely made no reference to the cordial for 4 glasses when answering this item. For the second response of 50ml, they demonstrated the correct ratio of syrup and water for one, two, three and four cups in a table, but then maybe used the same thinking as previously described when recording their incorrect response (Figure 1).

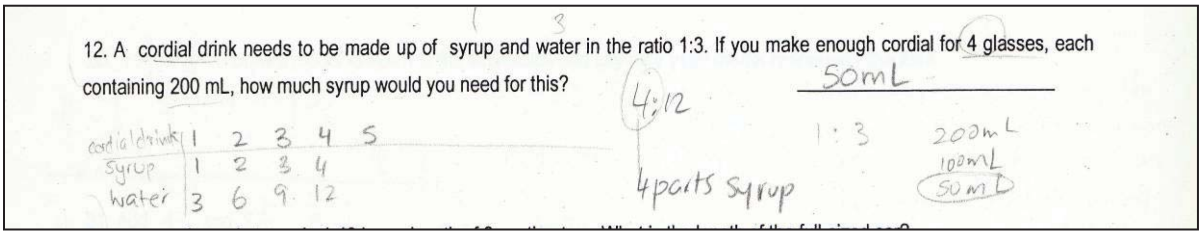
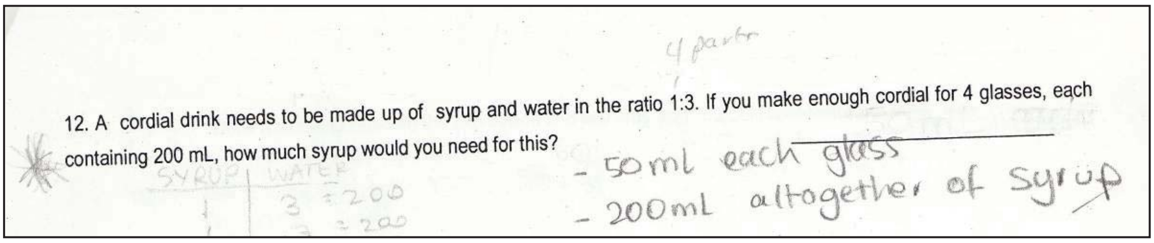


Figure 1. Example of second-year pre-service teacher incorrect response for Item 12.

Nearly one-third (30%) of pre-service teachers recorded a range of various other responses and lacked correct proportional reasoning strategies for Item 12. Most likely, a few (11%) pre-service teachers found this item very difficult and were unable to attempt the answer, leaving it blank. One pre-service teacher had recorded an answer but it was rubbed out.

Ratio is linked to multiplication (Reys et al., 2012; Suggate et al., 2006)). It can be concluded from their working out that the pre-service teachers who responded with 4ml and 50ml had some understanding of ratio, but most likely they were unable to interpret and use all of the numbers within the problem. They could not draw on their multiplicative thinking to correctly solve this comparison of liquid item.

When writing their correct responses, some pre-service teachers recorded their working out. For Item 12, there were two common correct methods used for calculating the correct answer. Two correct examples for Item 12 are shown, in Figure 2 and Figure 3. The first pre-service teacher used a table (Figure 3) and then possibly completed a multiplication of 4 by 50ml in their head. The second pre-service teacher (Figure 3) most likely used a division process to calculate the 50 ml, and then multiplied by 4.

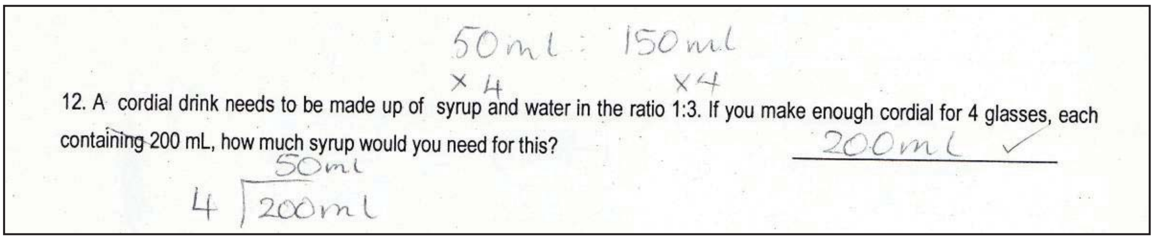

4 parts

12. A cordial drink needs to be made up of syrup and water in the ratio 1:3. If you make enough cordial for 4 glasses, each containing 200 mL, how much syrup would you need for this?

SYRUP | WATER
1 | 3 = 200
1 | 3 = 200

- 50ml each glass
- 200mL altogether of syrup

Figure 2. Example of second-year pre-service teacher correct response for Item 12.


50ml : 150ml
× 4 ×4

12. A cordial drink needs to be made up of syrup and water in the ratio 1:3. If you make enough cordial for 4 glasses, each containing 200 mL, how much syrup would you need for this? 200ml ✓

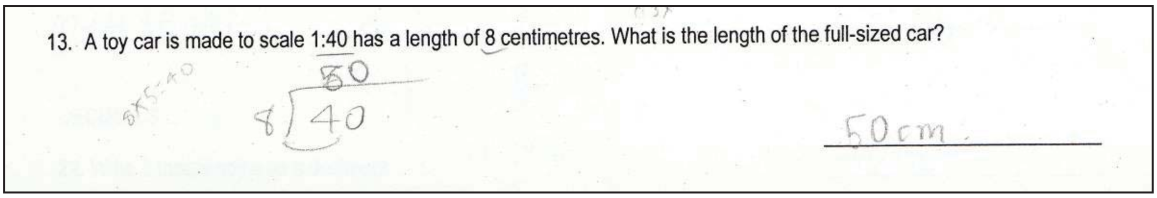
50ml
4 ⟌200ml

Figure 3. Example of second-year pre-service teacher correct response for Item 12.

Item 13 comparison of length

For Item 13 (see Table 3), there were no patterns of incorrect responses. A few pre-service teachers (9%) had various other answers, for example, recording the length of a full size car as too small 50 cm and 240 cm or too large 5 m. When thinking about the relationship of these numbers and the size of a toy car compared to a full size car, these pre-service teachers did not draw on number sense or likely correct responses. They most likely made an error with a "rule" when attempting to calculate the difference in scale.

Figure 4 (Item 13) is an example of a *various other* response. They attempted to divide 40 by 8 and incorrectly recorded 50, unable to correctly calculate a short division or number fact, and also an incorrect method for solving this item. They possibly lacked knowledge of division and multiplication and made an error with a "rule".


13. A toy car is made to scale 1:40 has a length of 8 centimetres. What is the length of the full-sized car?

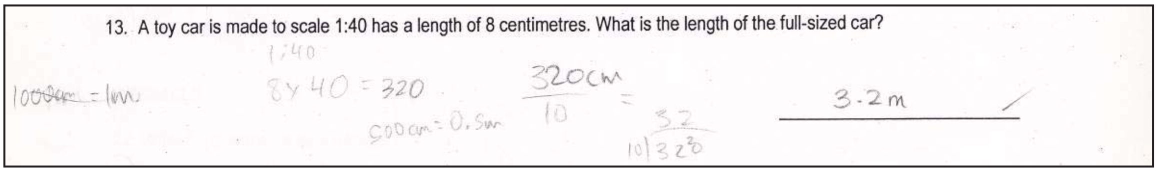
50
8⟌40

50cm

Figure 4. Example of second-year pre-service teacher incorrect response for Item 13.

Figure 5 shows this pre-service teacher was able to respond correctly to the comparison of length or scale item, providing detailed working out. This item required a comparison of measures and the numbers within the problem combined with the pre-service teacher's knowledge of multiplication facts may have made the item easy for him. The written response also demonstrated he knew 500 cm was equal to 0.5 m and 320 cm can be represented as 3.2 m. Of interest, when renaming 320 cm he recorded a division by ten, which should be 100 as 100 cm is equal to 1 m. However, he did record the correct response.

For Item 13, some other pre-service teachers were also able to convert their answer to 3.2 m but more chose to record their answer as 320 cm. Both responses were accepted as correct.


13. A toy car is made to scale 1:40 has a length of 8 centimetres. What is the length of the full-sized car?

1:40

1000cm = 1m

8 x 40 = 320

500cm = 0.5m

320cm / 10 =

3.2 / 10|320

3.2m

Figure 5. Example of second-year pre-service teacher misconception for Item 13.

Conclusion

All of the pre-service teachers in this study had completed half their program, that is, three mathematics education units of study and a range of mathematics experiences when participating in primary school visits. So it is surprising that many had difficulties related to interpreting the proportion item as well as demonstrating a lack of knowledge of multiplicative thinking, division and multiplication skills. Even the scale item caused difficulty for some pre-service teachers. The part-whole thinking needed in the scale item seemed to cause some issues for second-year, pre-service teachers even though they could respond correctly, demonstrating a lack of fluency in their proportional reasoning strategies. Both items relied on qualitative reasoning as the pre-service teachers had to interpret a written problem as noted by Heller et al. (1989) as the first step in the development of proportional reasoning. These pre-service teachers in this study did not use addition or subtraction (Suggate et al., 2006) to calculate the missing part, so demonstrated awareness that the method of solution for these items required multiplicative thinking. However, there were examples in their working out demonstrating errors in execution of the algorithms.

The Anchor Points were significant in facilitating the analysis of the responses of the two items and identifying the more difficult item of these two items. Anchor points have the potential to pinpoint the level of difficulty of items in tests assessing pre-service teachers' MCK. Consideration of Anchor Points may provide lecturers with benchmarks regarding the level of pre-service teachers' MCK so that an appropriate learning trajectory may assist their preparation for primary mathematics teaching.

Pre-service teachers should experience a range of problems and share their methods of solutions for similar proportion and ratio problems for discussion. This would build their MCK for solving these problems using more than one method and may assist pre-service teachers to consider the different methods from a conceptual focus rather than rely on a rule (Tall, 1997). They should be encouraged to model answers with pictures, look for patterns or in tables as this would develop their MCK and also assist their pedagogical content knowledge for teaching this topic to their students in the future.

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